1.2 ROW REDUCTION AND ECHelon FORMS

In this section, we refine the method of Section 1.1 into a row reduction algorithm that will enable us to analyze any system of linear equations.\(^1\) By using only the first part of the algorithm, we will be able to answer the fundamental existence and uniqueness questions posed in Section 1.1.

The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system. So the first part of this section concerns an arbitrary rectangular matrix. We begin by introducing two important classes of matrices that include the “triangular” matrices of Section 1.1. In the definitions that follow, a \textit{nonzero} row or column in a matrix means a row or column that contains at least one nonzero entry; a \textit{leading entry} of a row refers to the leftmost nonzero entry (in a nonzero row).

**DEFINITION**

A rectangular matrix is in \textbf{echelon form} (or \textbf{row echelon form}) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in \textbf{reduced echelon form} (or \textbf{reduced row echelon form}):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

An \textbf{echelon matrix} (respectively, \textbf{reduced echelon matrix}) is one that is in echelon form (respectively, reduced echelon form). Property 2 says that the leading entries form an \textit{echelon} ("steplike") pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.

The “triangular” matrices of Section 1.1, such as

\[
\begin{bmatrix}
2 & -3 & 2 & 1 \\
0 & 1 & -4 & 8 \\
0 & 0 & 0 & 5/2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

\(^1\)Our algorithm is a variant of what is commonly called \textit{Gaussian elimination}. A similar elimination method for linear systems was used by Chinese mathematicians in about 250 B.C. The process was unknown in Western culture until the nineteenth century, when a famous German mathematician, Carl Friedrich Gauss, discovered it. A German engineer, Wilhelm Jordan, popularized the algorithm in an 1888 text on geodesy.
are in echelon form. In fact, the second matrix is in reduced echelon form. Here are additional examples.

**EXAMPLE 1** The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries (*) may have any values (including zero).

\[
\begin{bmatrix}
\text{■} & \ast & \ast & \ast \\
0 & \text{■} & \ast & \ast \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & \text{■} & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & \text{■} & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{■} & \ast \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{■} & \ast
\end{bmatrix}
\]

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

\[
\begin{bmatrix}
1 & 0 & \ast & \ast \\
0 & 1 & \ast & \ast \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & \ast & 0 & 0 & 0 & \ast & \ast & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \ast & \ast & 0 \\
0 & 0 & 0 & 0 & 1 & \ast & \ast & \ast & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Any nonzero matrix may be row reduced (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique. The following theorem is proved in Appendix A at the end of the text.

**THEOREM 1**

**Uniqueness of the Reduced Echelon Form**

Each matrix is row equivalent to one and only one reduced echelon matrix.

If a matrix \( A \) is row equivalent to an echelon matrix \( U \), we call \( U \) an echelon form (or row echelon form) of \( A \); if \( U \) is in reduced echelon form, we call \( U \) the reduced echelon form of \( A \). [Most matrix programs and calculators with matrix capabilities use the abbreviation RREF for reduced (row) echelon form. Some use REF for (row) echelon form.]

**Pivot Positions**

When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries. Since the reduced echelon form is unique, the leading entries are always in the same positions in any echelon form obtained from a given matrix. These leading entries correspond to leading 1's in the reduced echelon form.
A **pivot position** in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the reduced echelon form of $A$. A **pivot column** is a column of $A$ that contains a pivot position.

In Example 1, the squares (●) identify the pivot positions. Many fundamental concepts in the first four chapters will be connected in one way or another with pivot positions in a matrix.

**Example 2** Row reduce the matrix $A$ below to echelon form, and locate the pivot columns of $A$.

$$A = \begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}$$

**Solution** Use the same basic strategy as in Section 1.1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

$$\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
0 & -3 & -6 & 4 & 9
\end{bmatrix}$$

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely, in the second column. We'll choose the 2 in this position as the next pivot.

$$\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 5 & 10 & -15 & -15 \\
0 & -3 & -6 & 4 & 9
\end{bmatrix} \quad (1)$$

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & 0
\end{bmatrix} \quad (2)$$
The matrix in (2) is different from any encountered in Section 1.1. There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
* & * & * & * & * \\
0 & * & * & * & * \\
0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of \( A \) are pivot columns.

\[
A = \begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}
\]

A pivot, as illustrated in Example 2, is a nonzero number in a pivot position that is used as needed to create zeros via row operations. The pivots in Example 2 were 1, 2, and \(-5\). Notice that these numbers are not the same as the actual elements of \( A \) in the highlighted pivot positions shown in (3). In fact, a different sequence of row operations might involve a different set of pivots. Also, a pivot will not be visible in the echelon form if the row is scaled to change the pivot to a leading 1 (which is often convenient for hand computations).

With Example 2 as a guide, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix. Careful study and mastery of the procedure now will pay rich dividends later in the course.

**The Row Reduction Algorithm**

The algorithm that follows consists of four steps, and it produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form. We illustrate the algorithm by an example.

**EXAMPLE 3** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & -5 \\
3 & -7 & 8 & -5 & 8 & 9 \\
3 & -9 & 12 & -9 & 6 & 15
\end{bmatrix}
\]
Solution

**STEP 1**
Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

\[
\begin{bmatrix}
0 & 3 & -6 & 6 & 4 & -5 \\
3 & -7 & 8 & -5 & 8 & 9 \\
3 & -9 & 12 & -9 & 6 & 15 \\
\end{bmatrix}
\]

**STEP 2**
Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
3 & -7 & 8 & -5 & 8 & 9 \\
0 & 3 & -6 & 6 & 4 & -5 \\
\end{bmatrix}
\]

**STEP 3**
Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 3 & -6 & 6 & 4 & -5 \\
\end{bmatrix}
\]

**STEP 4**
Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, we'll select as a pivot the "top" entry in that column.
For step 3, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the “top” row to the row below. This produces

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

**STEP 5**

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 0 & -9 \\
0 & 2 & -4 & 4 & 0 & -14 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix} \quad \text{Row } 1 + (-6) \cdot \text{row } 3
\]

\[
\begin{bmatrix}
0 & 2 & -4 & 4 & 0 & -14 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix} \quad \text{Row } 2 + (-2) \cdot \text{row } 3
\]

The next pivot is in row 2. Scale this row, dividing by the pivot.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 0 & -9 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix} \quad \text{Row scaled by } \frac{1}{2}
\]

Create a zero in column 2 by adding 9 times row 2 to row 1.

\[
\begin{bmatrix}
3 & 0 & -6 & 9 & 0 & -72 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix} \quad \text{Row } 1 + (9) \cdot \text{row } 2
\]
Finally, scale row 1, dividing by the pivot, 3.

\[
\begin{bmatrix}
1 & 0 & -2 & 3 & 0 & -24 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\leftarrow \text{Row scaled by } \frac{1}{3}
\]

This is the reduced echelon form of the original matrix.

The combination of steps 1–4 is called the forward phase of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the backward phase.

**Numerical Note**

In step 2 above, a computer program usually selects as a pivot the entry in a column having the largest absolute value. This strategy, called partial pivoting, is used because it reduces roundoff errors in the calculations.

**Solutions of Linear Systems**

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form

\[
\begin{bmatrix}
1 & 0 & -5 & 1 \\
0 & 1 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

There are three variables because the augmented matrix has four columns. The associated system of equations is

\[
\begin{align*}
  x_1 - 5x_3 &= 1 \\
  x_2 + x_3 &= 4 \\
  0 &= 0
\end{align*}
\]

The variables \(x_1\) and \(x_2\) corresponding to pivot columns in the matrix are called basic variables.\(^2\) The other variable, \(x_3\), is called a free variable.

Whenever a system is consistent, as in (4), the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

\(^2\)Some texts use the term *leading variables* because they correspond to the columns containing leading entries.
variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation. In (4), we can solve the first equation for \( x_1 \) and the second for \( x_2 \). (The third equation is ignored; it offers no restriction on the variables.)

\[
\begin{align*}
  x_1 &= 1 + 5x_3 \\
  x_2 &= 4 - x_3 \\
  x_3 &\text{ is free}
\end{align*}
\]  

(5)

By saying that \( x_3 \) is “free,” we mean that we are free to choose any value for \( x_3 \). Once that is done, the formulas in (5) determine the values for \( x_1 \) and \( x_2 \). For instance, when \( x_3 = 0 \), the solution is \((1, 4, 0)\); when \( x_3 = 1 \), the solution is \((6, 3, 1)\). Each different choice of \( x_3 \) determines a (different) solution of the system, and every solution of the system is determined by a choice of \( x_3 \).

The solution in (5) is called a \textbf{general solution} of the system because it gives an explicit description of all solutions.

\section*{Example 4}

Find the general solution of the linear system whose augmented matrix has been reduced to

\[
\begin{bmatrix}
  1 & 6 & 2 & -5 & -2 & -4 \\
  0 & 0 & 2 & -8 & -1 & 3 \\
  0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix}
\]

\section*{Solution}

The matrix is in echelon form, but we want the reduced echelon form before solving for the basic variables. The row reduction is completed next. The symbol \( \sim \) before a matrix indicates that the matrix is row equivalent to the preceding matrix.

\[
\begin{bmatrix}
  1 & 6 & 2 & -5 & -2 & -4 \\
  0 & 0 & 2 & -8 & -1 & 3 \\
  0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 6 & 2 & -5 & -2 & -4 \\
  0 & 0 & 0 & 0 & 1 & 7 \\
  0 & 0 & 0 & 1 & -4 & 5 \\
  0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 6 & 2 & -5 & 0 & 10 \\
  0 & 0 & 1 & -4 & 0 & 5 \\
  0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix}
\]

There are five variables because the augmented matrix has six columns. The associated system now is

\[
\begin{align*}
  x_1 + 6x_2 &+ 3x_4 = 0 \\
  x_3 - 4x_4 &= 5 \\
  x_5 &= 7
\end{align*}
\]  

(6)

The pivot columns of the matrix are 1, 3, and 5, so the basic variables are \( x_1, x_3, \) and \( x_5 \). The remaining variables, \( x_2 \) and \( x_4 \), must be free. Solving for the basic variables,
we obtain the general solution:

\[
\begin{align*}
    x_1 &= -6x_2 - 3x_4 \\
    x_2 &= \text{free} \\
    x_3 &= 5 + 4x_4 \\
    x_4 &= \text{free} \\
    x_5 &= 7
\end{align*}
\]  

(7)

Note that the value of \( x_5 \) is already fixed by the third equation in system (6).

**Parametric Descriptions of Solution Sets**

The descriptions in (5) and (7) are parametric descriptions of solution sets in which the free variables act as parameters. Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. For instance, in system (4), we may add 5 times equation 2 to equation 1 and obtain the equivalent system

\[
\begin{align*}
    x_1 + 5x_2 &= 21 \\
    x_2 + x_3 &= 4
\end{align*}
\]

We could treat \( x_2 \) as a parameter and solve for \( x_1 \) and \( x_3 \) in terms of \( x_2 \), and we would have an accurate description of the solution set. However, to be consistent, we make the (arbitrary) convention of always using the free variables as the parameters for describing a solution set. (The answer section at the end of the text also reflects this convention.)

Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.

**Back-Substitution**

Consider the following system, whose augmented matrix is in echelon form but is not in reduced echelon form:

\[
\begin{align*}
    x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\
    x_2 - 3x_3 + 3x_4 + x_5 &= -5 \\
    x_4 - x_5 &= 4
\end{align*}
\]

A computer program would solve this system by back-substitution, rather than by computing the reduced echelon form. That is, the program would solve equation 3 for \( x_4 \) in terms of \( x_5 \) and substitute the expression for \( x_4 \) into equation 2, solve equation 2 for \( x_2 \), and then substitute the expressions for \( x_2 \) and \( x_4 \) into equation 1 and solve for \( x_1 \).

Our matrix format for the backward phase of row reduction, which produces the reduced echelon form, has the same number of arithmetic operations as back-substitution. But the discipline of the matrix format substantially reduces the likelihood of errors.
during hand computations. I strongly recommend that you use only the reduced echelon form to solve a system! The Study Guide that accompanies this text offers several helpful suggestions for performing row operations accurately and rapidly.

**Numerical Note**

In general, the forward phase of row reduction takes much longer than the backward phase. An algorithm for solving a system is usually measured in flops (or floating point operations). A flop is one arithmetic operation (+, −, *, /) on two real floating point numbers.\(^3\) For an \(n \times (n+1)\) matrix, the reduction to echelon form can take \(2n^3/3 + n^2/2 - 7n/6\) flops (which is approximately \(2n^3/3\) flops when \(n\) is moderately large—say, \(n \geq 30\)). In contrast, further reduction to reduced echelon form needs at most \(n^2\) flops.

**Existence and Uniqueness Questions**

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental questions posed in Section 1.1.

**Example 5** Determine the existence and uniqueness of the solutions to the system

\[
\begin{align*}
3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\
3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\
3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15
\end{align*}
\]

**Solution** The augmented matrix of this system was row reduced in Example 3 to

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

The basic variables are \(x_1, x_2,\) and \(x_5;\) the free variables are \(x_3\) and \(x_4.\) There is no equation such as 0 = 1 that would create an inconsistent system, so we could use back-substitution to find a solution. But the existence of a solution is already clear in (8). Also, the solution is not unique because there are free variables. Each different choice of \(x_3\) and \(x_4\) determines a different solution. Thus the system has infinitely many solutions.

\(^3\) Traditionally, a flop was only a multiplication or division, because addition and subtraction took much less time and could be ignored. The definition of flop given here is preferred now, as a result of advances in computer architecture. See Golub and Van Loan, *Matrix Computations*, 2nd ed. (Baltimore: The Johns Hopkins Press, 1989), pp. 19-20.
When a system is in echelon form and contains no equation of the form \(0 = b\), with \(b\) nonzero, every nonzero equation contains a basic variable with a nonzero coefficient. Either the basic variables are completely determined (with no free variables) or at least one of the basic variables may be expressed in terms of one or more free variables. In the former case, there is a unique solution; in the latter case, there are infinitely many solutions (one for each choice of values for the free variables).

These remarks justify the following theorem.

**THEOREM 2**

**Existence and Uniqueness Theorem**

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

\[
\begin{bmatrix}
0 & \cdots & 0 & b
\end{bmatrix}
\]

with \(b\) nonzero

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.

**USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM**

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

**PRACTICE PROBLEMS**

1. Find the general solution of the linear system whose augmented matrix is

\[
\begin{bmatrix}
1 & -3 & -5 & 0 \\
0 & 1 & 1 & 3
\end{bmatrix}
\]
2. Find the general solution of the system

\[
{x_1} - 2{x_2} - {x_3} + 3{x_4} = 0
\]
\[
-2{x_1} + 4{x_2} + 5{x_3} - 5{x_4} = 3
\]
\[
3{x_1} - 6{x_2} - 6{x_3} + 8{x_4} = 2
\]

1.2 Exercises

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 3 & 3 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}
\]

2. a. \[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. \[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
4 & 5 & 6 & 7 \\
6 & 7 & 8 & 9
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
1 & 3 & 5 & 7 \\
3 & 5 & 7 & 9 \\
5 & 7 & 9 & 1
\end{bmatrix}
\]

5. Describe the possible echelon forms of a nonzero 2 × 2 matrix. Use the symbols \(\neq\), \(*\), and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero 3 × 2 matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. \[
\begin{bmatrix}
1 & 4 & 0 & 7 \\
3 & 9 & 7 & 6 \\
0 & 1 & -6 & 5
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 4 & 0 & 7 \\
2 & 7 & 0 & 10 \\
1 & -2 & -1 & 3
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
3 & -4 & 2 & 0 \\
-9 & 12 & -6 & 0 \\
-6 & 8 & -4 & 0
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
1 & -7 & 0 & 6 & 5 \\
0 & 0 & 1 & -2 & -3 \\
-1 & 7 & -4 & 2 & 7
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
1 & -3 & 0 & -1 & 0 & -2 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
1 & 2 & -5 & -6 & 0 & -5 \\
0 & 1 & -6 & -3 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Exercises 15 and 16 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

15. a. \[
\begin{bmatrix}
\neq & * & * & * \\
0 & \neq & * & * \\
0 & 0 & \neq & 0
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
0 & \neq & * & * & * \\
0 & 0 & \neq & * & * \\
0 & 0 & 0 & 0 & \neq
\end{bmatrix}
\]
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CHAPTER 1 Linear Equations in Linear Algebra

16. a. \[
\begin{bmatrix}
0 & * & * \\
0 & 0 & 0 \\
\end{bmatrix}
\]
b. \[
\begin{bmatrix}
* & * & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & * & * \\
\end{bmatrix}
\]

In Exercises 17 and 18, determine the value(s) of \( h \) such that the matrix is the augmented matrix of a consistent linear system.

17. \[
\begin{bmatrix}
2 & 3 & h \\
4 & 6 & 7 \\
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
1 & -3 & -2 \\
5 & h & -7 \\
\end{bmatrix}
\]

In Exercises 19 and 20, choose \( h \) and \( k \) such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

19. \( x_1 + hx_2 = 2 \)

\( 4x_1 + 8x_2 = k \)

20. \( x_1 + 3x_2 = 2 \)

\( 3x_1 + hx_2 = k \)

In Exercises 21 and 22, mark each statement True or False. Justify each answer.⁴

21. a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

b. The row reduction algorithm applies only to augmented matrices for a linear system.

c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

d. Finding a parametric description of the solution set of a linear system is the same as solving the system.

e. If one row in an echelon form of an augmented matrix is \([0 \ 0 \ 0 \ 5 \ 0]\), then the associated linear system is inconsistent.

22. a. The echelon form of a matrix is unique.

b. The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.

c. Reducing a matrix to echelon form is called the \textit{forward phase} of the row reduction process.

d. Whenever a system has free variables, the solution set contains many solutions.

e. A general solution of a system is an explicit description of all solutions of the system.

23. Suppose a \(3 \times 5\) \textit{coefficient} matrix for a system has three pivot columns. Is the system consistent? Why or why not?

24. Suppose a system of linear equations has a \(3 \times 5\) \textit{augmented} matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)?

25. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

26. Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot in each column. Explain why the system has a unique solution.

27. Restate the last sentence in Theorem 2 using the concept of pivot columns: \textit{"If a linear system is consistent, then the solution is unique if and only if...".}

28. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

29. A system of linear equations with fewer equations than unknowns is sometimes called an \textit{underdetermined system}. Suppose that such a system happens to be consistent. Explain why there must be an infinite number of solutions.

30. Give an example of an inconsistent underdetermined system of two equations in three unknowns.

31. A system of linear equations with more equations than unknowns is sometimes called an \textit{overdetermined system}. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.

32. Suppose an \(n \times (n + 1)\) matrix is row reduced to reduced echelon form. Approximately what fraction of the total number of operations (flops) is involved in the backward phase of the reduction when \(n = 30\)? when \(n = 300\)?

Suppose experimental data are represented by a set of points in the plane. An \textit{interpolating polynomial} for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.

⁴True/false questions of this type will appear in many sections. Methods for justifying your answers were described before Exercises 23 and 24 in Section 1.1.
33. Find the interpolating polynomial \( p(t) = a_0 + a_1 t + a_2 t^2 \) for the data \((1, 12), (2, 15), (3, 16)\). That is, find \( a_0, a_1, \) and \( a_2 \) such that

\[
\begin{align*}
    a_0 + a_1(1) + a_2(1)^2 &= 12 \\
    a_0 + a_1(2) + a_2(2)^2 &= 15 \\
    a_0 + a_1(3) + a_2(3)^2 &= 16
\end{align*}
\]

34. [M] In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

<table>
<thead>
<tr>
<th>Velocity (100 ft/sec)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force (100 lb)</td>
<td>0</td>
<td>2.90</td>
<td>14.8</td>
<td>39.6</td>
<td>74.3</td>
<td>119</td>
</tr>
</tbody>
</table>

Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile is traveling at 750 ft/sec. Use \( p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \).

What happens if you try to use a polynomial of degree less than 5? (Try a cubic polynomial, for instance.)

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Exercises marked with the symbol [M] are designed to be worked with the aid of a “Matrix program” (a computer program, such as MATLAB, Maple, Mathematica, MathCad, or Derive, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments or Hewlett-Packard).